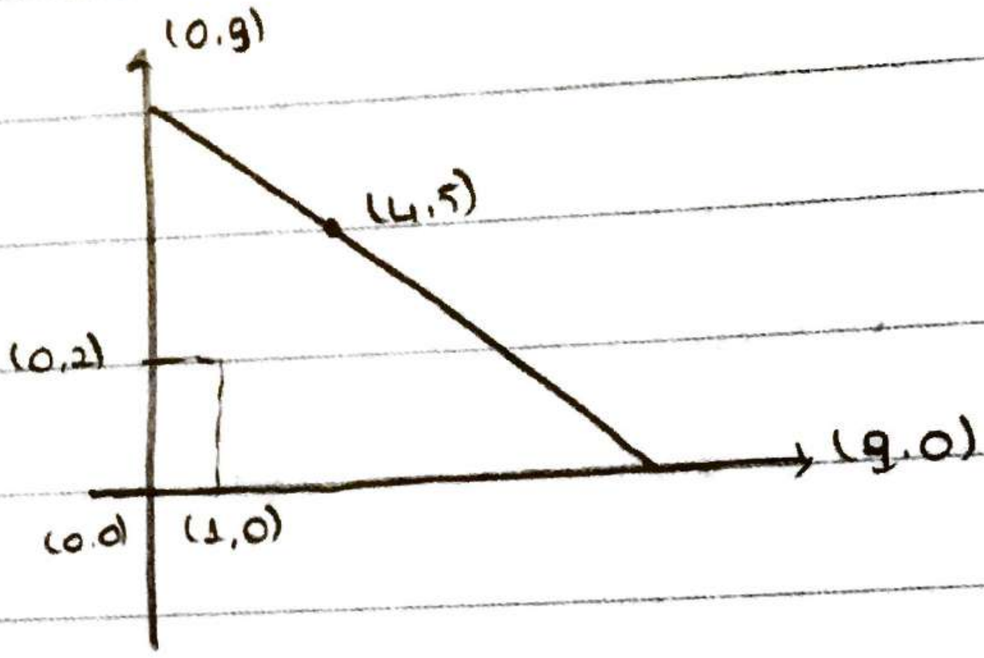


$f(0,0) = 2$

$f(1,0) = 3$

$f(9,0) = -61 \rightarrow$ abs. min

$f(0,2) = 4$

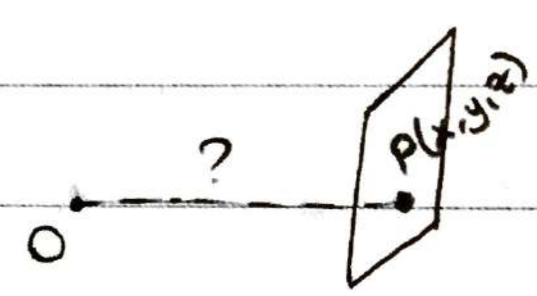
$f(1,2) = 7 \rightarrow$ abs. max

$f(4,4) = -11$

$f(0,9) = -45$

14.8 → LAGRANGE MULTIPLIERS

ex: Find the point on the plane $2x+y-z-5=0$ that is closest to the origin.



$d(x,y,z)$ = distance between (x,y,z) and $(0,0,0)$

$= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$= \sqrt{x^2+y^2+z^2}$

min $d(x,y,z)$ on $2x+y-z-5=0$

$\downarrow$

$z = 2x+y-5$

* First way

Find minimum of $f(x,y) = d(x,y,2x+y-5)$

$= \sqrt{x^2+y^2+(2x+y-5)^2}$ ★

$f_x = \frac{1}{2\sqrt{★}}(2x + 2(2x+y-5).2) = 0 \Rightarrow 2x + 8x + 4y - 20 = 0$

$f_y = \frac{1}{2\sqrt{★}}(2y + 2(2x+y-5)) = 0 \Rightarrow 2y + 4x + 2y - 10 = 0$

$10x + 4y = 20$

$-/ \; 4x + 4y = 10$

$6x = 10 \rightarrow x = 5/3 \quad y = 5/6$

$z = 2 . 5/3 + 5/6 + 5 = -5/6$

$P(5/3, 5/6, -5/6) \rightarrow$ is the point closest to origin on the plane.

Remark Alternative method : search for minimum of the function

$$g(x,y,z) = x^2 + y^2 + z^2$$

instead of

$$d(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

* Second way:

- The Method of Lagrange Multipliers

Find the min / max of $f(x,y,z)$ subject to the constraint

$$g(x,y,z) = 0$$

The min / max points satisfy $(\lambda \in \mathbb{R})$

$$\nabla f = \lambda . \nabla g \iff \begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda . \dfrac{\partial g}{\partial y} \\ \dfrac{\partial f}{\partial z} = \lambda . \dfrac{\partial g}{\partial z} \end{cases}$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k$$

⇒ Second method for previous problem

constraint : $g(x,y,z) = 2x+y-z-5=0$

optimize : $f(x,y,z) = x^2+y^2+z^2 = 0$

$$\frac{\partial f}{\partial x} = 2x = \lambda . \frac{\partial g}{\partial x} = \lambda . 2$$

$$\frac{\partial f}{\partial y} = 2y = \lambda . \frac{\partial g}{\partial y} = \lambda$$

$$\frac{\partial f}{\partial z} = 2z = \lambda . \frac{\partial g}{\partial z} = -\lambda$$

$2x+y-z=5$

$2x = 2\lambda \longrightarrow x = \lambda$

$2y = \lambda \longrightarrow y = \lambda/2$

$2z = -\lambda \longrightarrow z = -\lambda/2$

$2\lambda + \lambda/2 + \lambda/2 = 5$

$3\lambda = 5$

$\lambda = 5/3$

$x = 5/3 \quad y = 5/6 \quad z = -5/6$

ex: Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse

$$\boxed{\frac{x^2}{8} + \frac{y^2}{2} = 1}$$

$\nabla f = \lambda \nabla g$

$g(x,y) = 0$

$\frac{\partial f}{\partial x} = y = \lambda . \frac{\partial g}{\partial x} = \lambda . \frac{x}{4} \rightarrow \boxed{4y = \lambda . x}$

$\frac{\partial f}{\partial y} = x = \lambda . \frac{\partial g}{\partial x} = \lambda . y \rightarrow \boxed{x = \lambda . y}$

$$\begin{cases} 4y = \lambda x \\ x = \lambda y \\ \dfrac{x^2}{8} + \dfrac{y^2}{2} = 1 \end{cases} \quad \left.\begin{matrix} 4y = \lambda x \\ x = \lambda y \end{matrix}\right\} 4y = \lambda^2 y$$

$4y = \lambda^2 y \Rightarrow y = 0 \Rightarrow x = 0$ NOT A SOLUTION!

$\Rightarrow y \neq 0 \Rightarrow \lambda^2 = 4$

$\lambda = 2$ $\qquad$ $\lambda = -2$

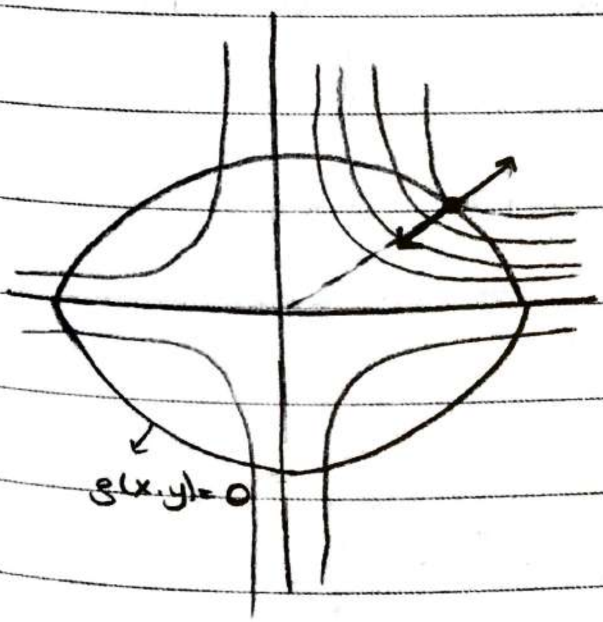
$x = 2y$ $\qquad$ $x = -2y$

$\frac{y^2}{2} + \frac{y^2}{2} - 1 = 0$ $\qquad$ $\frac{y^2}{2} + \frac{y^2}{2} - 1 = 0$

$y = \mp 1$ $\qquad$ $y = \mp 1$

$(x,y) = (2,1), (-2,-1)$ $\qquad$ $(x,y) = (-2,1)$ or $(2,-1)$

- Geometry of the last example



$xy = f(x,y) = \text{constant}$

ex: Find the max/min of $f(x,y) = 3x+4y$ on the circle

$$g(x,y) = x^2+y^2-1=0$$

$$\frac{\partial f}{\partial x} = 3 = \frac{\partial g}{\partial x} = \lambda . 2x \longrightarrow 3 = \lambda . 2x$$

$$\frac{\partial f}{\partial y} = 4 = \frac{\partial g}{\partial y} = \lambda . 2y \longrightarrow 4 = \lambda . \cancel{2}y$$

$$\left\{\begin{array}{l} 3 = 2x . \lambda \\ 4 = 2y . \lambda \\ x^2+y^2-1=0 \end{array}\right. \quad 3\overset{\overset{4m}{\uparrow}}{y} = 4\overset{\overset{3m}{\uparrow}}{x}$$

$$9m^2+16m^2=1$$

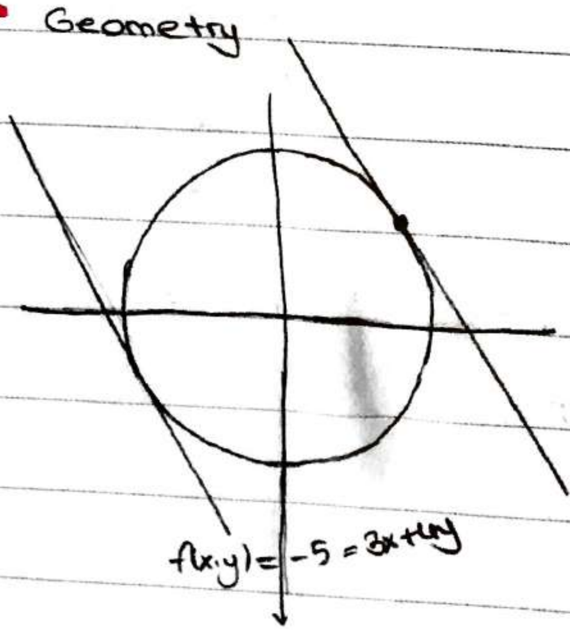
$$25m^2=1$$

$m = 1/5$	$m = -1/5$
$x = 3/5$	$x = -3/5$
$y = 4/5$	$y = -4/5$
$(3/5, 4/5)$	$(-3/5, -4/5)$

$f(3/5, 4/5) = 5 \rightarrow$ max

$f(-3/5, -4/5) = -5 \rightarrow$ min

- Geometry



Langrange multipliers with two constraints:

Find min/max of $f(x,y,z)$

Two constraints

$$g_1(x,y,z) = 0$$

$$g_2(x,y,z) = 0$$

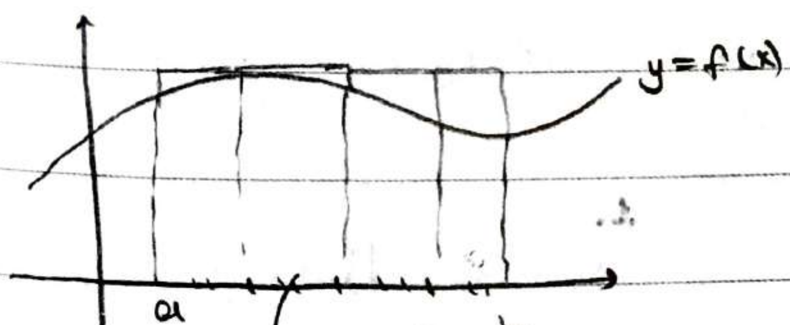
$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

ex: Look at the book.

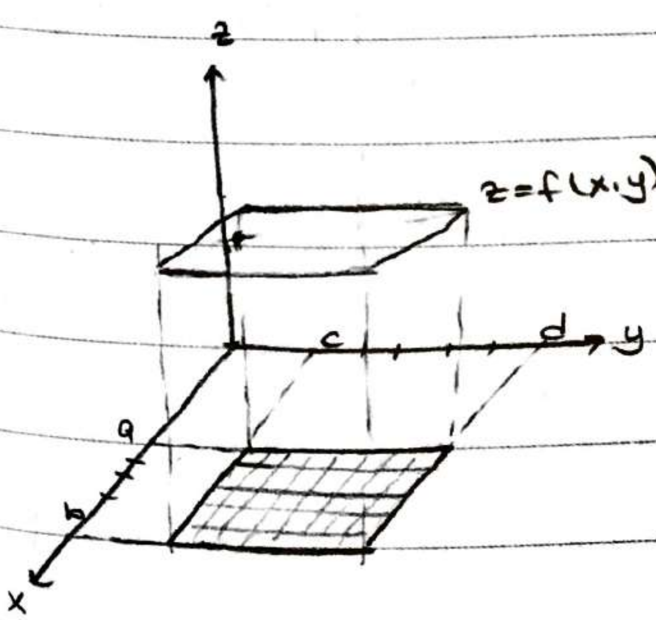
CHAPTER 15 - MULTIPLE INTEGRALS

15.1 → DOUBLE INTEGRAL OVER RECTANGLES

Recall Integrals of One-Variable Function.



$$\text{Area} = \int_a^b f(x)\,dx$$



$$\text{Volume} = \iint_{[a,b]\times[c,d]} f(x,y)\,dA$$

Fubinis Theorem:

Let $f(x,y)$ be a continuous function on the rectangle,

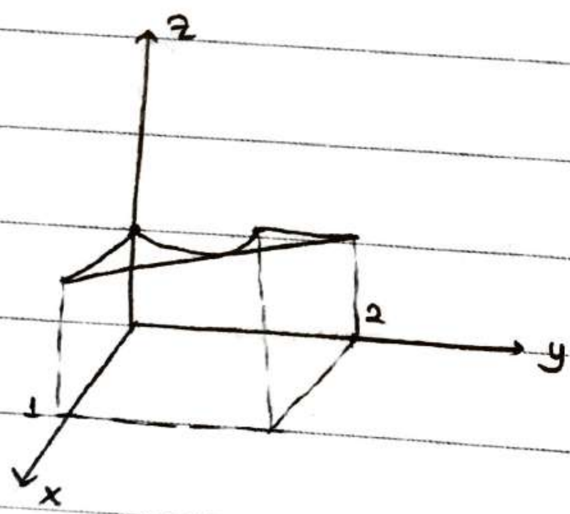
$$R = [a,b] \times [c,d]$$
$$= \{(x,y) : a \leq x \leq b \ , \ c \leq y \leq d\}$$

Then

$$\iint_{[a,b]\times[c,d]} f(x,y) \, dA = \int_{x=a}^{b} \left[\int_{c}^{d} f(x,y)\,dy \right] dx$$

$$= \int_{y=c}^{d} \left[\int_{a}^{b} f(x,y)\,dx \right] dy$$

ex: Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R $0 \leq x \leq 1$, $0 \leq y \leq 2$



1.

$$\text{Volume} = \iint_{[0,1]x[2x0]} (10 + x^2 + 3y^2)\,dA$$

$$= \int_{x=0}^{1} \left[\int_{0}^{2} (10 + x^2 + 3y^2)\,dy \right] dx$$

$$= \int_{0}^{1} \left[10y + x^2y + 3y^3 \Big|_{0}^{2} \right] dx$$

$$= \int_{0}^{1} [20 + 2x^2 + 8]\,dx$$

$$= 20x + \frac{2x^3}{3} + 8x \Big|_{0}^{1}$$

$$= 20 + \frac{2}{3} + 8 = 28\tfrac{2}{3}$$

$$\text{Volume} = \int_{y=0}^{2} \left[\int_{0}^{1} (10 + x^2 + 3y^2)\,dx \right] dy$$

$$= \int_{0}^{2} \left[10x + x^3/3 + 3xy^2 \Big|_{0}^{1} \right] dy$$

$$= 10y + \frac{y}{3} + y^3 \Big|_{0}^{2}$$

$$= 20 + \frac{2}{3} + 8 = 28\tfrac{2}{3}$$